The Chinese University of Hong Kong
Department of Mathematics
2024 Enrichment Programme for Young Mathematics Talents

TOWARDS DIFFERENTIAL GEOMETRY
Test 1, 16/08/2024

Instructions:

- Time allowed: **$90 \pm \delta$ minutes.**
- This paper consists of **Basic Part**, **Harder Part** and **Bonus Part**.
- The full mark of the paper is **80 points** and bonus mark **15 points**.
- Answer **ALL** questions in Basic Part and **ANY TWO** questions in Harder Part. Make your best effort to answer the Bonus Part.
- Show your work clearly and concisely. Give adequate explanation and justification for your calculation and observation.
- Write your answers in the spaces provided in the Answer Booklet.
- Calculators approved by the HKEAA are allowed.
- Supplementary answer sheets and rough paper will be supplied on request.
- Unless otherwise specified, numerical answers must be exact.

Full Name: ________________________ Group: ________________

Basic Part (50 points). Answer **ALL** questions in this part.

1. Let $\mathbf{u} = (a, -1, 0), \mathbf{v} = (b, -a, b), \mathbf{w} = (2, 1, -1)$. Suppose $\langle \mathbf{u}, (2, 2, 1) \rangle = 0$, $\|\mathbf{v}\| = 3$ and $b > 0$.
   (a) (3 points) Find a and b.
   (b) (3 points) Show that $\langle \mathbf{w}, \mathbf{u} \times \mathbf{v} \rangle \neq 0$.
   (c) (2 points) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.
   (d) (3 points) Find the volume of the tetrahedron generated by $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.
   (e) (3 points) Find the distance between $\mathbf{w}$ and the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

2. Give a parametrization $\mathbf{r}(t)$ of the following curves, specify the domain so that $\mathbf{r}(t)$ is regular.
   (a) (3 points) The line segment joining $(-5, 2)$ and $(2, 7)$.
   (b) (4 points) The circle of radius 3 centered at $(7, -5)$.
   (c) (4 points) The ellipse $\dfrac{(x-5)^2}{9} + \dfrac{(y-3)^2}{8} = 2$.
   (d) (4 points) The hyperbola $\dfrac{27x^2}{4} - \dfrac{y^2}{6} = 3$, where $x < 0$.

3. Let $\mathbf{r}(t) = \left(\tan t - t, \dfrac{\tan^2 t}{2} - \ln(\sec t) \right)$, $0 < t < \dfrac{\pi}{4}$, be a curve on the xy-plane.
   (a) (3 points) Show that $\mathbf{r}(t)$ is a regular parametrized curve.
   (b) (5 points) Provided that

$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \quad 2 \leq n \in \mathbb{N}.$$

   Find the arclength of $\mathbf{r}(t)$ from $t = 0$ to $t = \dfrac{\pi}{4}$.

4. (a) (2 points) Define $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$.
   Prove that $\cosh^2 x - \sinh^2 x = 1$ for any $x \in \mathbb{R}$.
   (b) (3 points) Denote $y = \sinh^{-1} x$. By considering $x = \dfrac{e^y - e^{-y}}{2}$, show that

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

   for any $x \in \mathbb{R}$.
   (c) Consider the curve **catenary** given by the local parametrization

$$\mathbf{r}(t) = (t, \cosh t), \quad t > 0.$$

   (i) (3 points) Compute $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.
   (ii) (5 points) Find the arclength parametrization of $\mathbf{r}(t)$. Leave your answer free of sinh and cosh functions.

Harder Part (30 points). Answer **ANY TWO** questions in this part.

5. Denote the transpose of a matrix M by M^T.
   (a) (5 points) Suppose that $\mathbf{u} \in \mathbb{R}^3$ and A is a 3×3 matrix. Show that
   $$\left\| \left(\frac{A + A^T}{2} \right) \mathbf{u} \right\|^2 + \left\| \left(\frac{A - A^T}{2} \right) \mathbf{u} \right\|^2 = \frac{1}{2} \|A\mathbf{u}\|^2 + \frac{1}{2} \|A^T\mathbf{u}\|^2 .$$
   (b) (5 points) Show that if $\left(\frac{A + A^T}{2} \right)^2 \mathbf{u} = \mathbf{0}$ for any $\mathbf{u} \in \mathbb{R}^3$, then
   $$A^T\mathbf{u} = -A\mathbf{u}.$$
   (c) (5 points) Someone claims that if $\left(\frac{A - A^T}{2} \right)^2 \mathbf{v} = \mathbf{0}$ for any $\mathbf{v} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, then A is symmetric. Is the claim correct? Explain your answer.

6. Fix a point $\mathbf{c} \in \mathbb{R}^3$. Suppose $\mathbf{r} : [a, b] \to \mathbb{R}^3$, be a regular *closed* curve, i.e. $\mathbf{r}(a) = \mathbf{r}(b)$, and it is maintaining a non-zero fixed distance from $\mathbf{c}$ for any $a \le t \le b$.
   (a) (3 points) Describe the shape of $\mathbf{r}(t)$.
   (b) (i) (3 points) Show that for any $a < t < b$,
   $$\langle \mathbf{r}(t) - \mathbf{c}, \mathbf{r}'(t) \rangle = 0.$$
   (ii) (3 points) Someone claims that for any regular curve $\mathbf{s} : [a, b] \to \mathbb{R}^3$ satisfying
   $$\langle \mathbf{s}(t) - \mathbf{c}, \mathbf{s}'(t) \rangle = 0$$
   for every $a < t < b$, then $\mathbf{s}(t)$ is a curve maintaining fixed distance from $\mathbf{c}$. Do you agree? Explain your answer.
   (c) (6 points) Suppose speed and magnitude of acceleration for $\mathbf{r}(t)$ are constant, i.e. $\|\mathbf{r}'(t)\|$ and $\|\mathbf{r}''(t)\|$ are constant. Given that $|\langle \mathbf{a}, \mathbf{b} \rangle|^2 + \|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2$, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and $\|\mathbf{r}(t) - \mathbf{c}\| \|\mathbf{r}''(t)\| > \|\mathbf{r}'(t)\|^2$. Prove that $(\mathbf{r}(t) - \mathbf{c}) \times \mathbf{r}''(t) = k\mathbf{r}'(t)$, for some $k \in \mathbb{R} \setminus \{0\}$.

7. Consider a curve $\mathbf{r} : (0, 1) \to \mathbb{R}^2$ parametrized by
   $$\mathbf{r}(t) = \left(t, \sin\left(\frac{1}{t} \right) \right)$$
   (a) (5 points) Is this curve regular? Is this curve parametrized by arc length?
   (b) (2 points) Fix a positive integer k, express $\mathbf{r}\left(\left(2k\pi + \frac{\pi}{2} \right)^{-1} \right)$ and $\mathbf{r}((2k\pi)^{-1})$ in terms of k.
   (c) (8 points) Given that the shortest curve between 2 points on $\mathbb{R}^2$ is a straight line. By considering the interval
   $$I_k \quad \text{defined by} \quad \left[\left(2k\pi + \frac{\pi}{2} \right)^{-1}, (2k\pi)^{-1} \right]$$
   for $k = 1, 2, 3 \cdots$, or otherwise, prove that $\mathbf{r}(t)$ has an infinite length over $(0, 1)$.

Bonus Part (15 points). Try your best to answer the question in this part. **Good luck**.

8. Consider a cycloid in the 2D plane which is parametrized by $(\theta - \sin\theta, 1 - \cos\theta)$ for $\theta \in (0, 2\pi)$. We now put it in $\mathbb{R}^3$ with $z = 0$ for all θ such that it lies on the xy-plane, i.e. its new parametrization is $(\theta - \sin\theta, 1 - \cos\theta, 0)$ for $\theta \in (0, 2\pi)$.
   Now we introduce the 3 basic rotation matrices in $\mathbb{R}^3$:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

$$R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   which rotate vectors by an angle α about the x-, y-, or z-axis, in three dimensions, using the right-hand rule, by multiplying these matrices to a vector from the left.
   (For example, to rotate about the x-axis, using right-hand rule means pointing your thumb towards the direction of x-axis, and the direction of rotation follows your other fingers)

   (a) (2 points) By picking a suitable rotational matrix and angle α, show that a cycloid lying on the xz-plane is parametrized by $\mathbf{r}(\theta) = (\theta - \sin\theta, 0, 1 - \cos\theta)$.

   (b) (2 points) Find $\mathbf{r}'(\theta)$ and $\|\mathbf{r}'(\theta)\|$.

   (c) (2 points) Find the angle $\varphi(\theta)$ between $\mathbf{e_3} = (0, 0, 1)$ and $\mathbf{r}'(\theta)$ in terms of θ.

   (d) Now also consider a helix parametrized by $(\cos\theta, \sin\theta, \theta)$ for $\theta \in (0, 2\pi)$. Imagine that now the helix rotates around $\mathbf{r}(\theta)$ instead of the z-axis. We will find a parametrization $\gamma(\theta)$ for $\theta \in (0, 2\pi)$ for that new curve in this part.
   Observe that
   $$(\cos\theta, \sin\theta, \theta) = (\cos\theta, \sin\theta, 0) + (0, 0, \theta).$$
   The helix is made up of two parts. Given that $\gamma(\theta)$ is also made up of two parts. Let $\mathbf{u}(\theta) = (\cos\theta, \sin\theta, 0)$ for $\theta \in (0, 2\pi)$.

      (i) (3 points) Rotate $\mathbf{u}(\theta)$ about the y-axis for angle $\varphi(\theta)$. Let the result be $\mathbf{v}(\theta)$. Show that $\mathbf{v}(\theta)$ satisfies the following property:
      $$\langle \mathbf{v}(\theta), \mathbf{r}'(\theta) \rangle = 0 \text{ for any } \theta \in (0, 2\pi)$$

      (ii) (1 point) Given that $\gamma(\theta) = \mathbf{v}(\theta) + \mathbf{r}(\theta)$. Show that $\gamma(\theta)$ satisfy the following:
      $$\langle \gamma(\theta) - \mathbf{r}(\theta), \mathbf{r}'(\theta) \rangle = 0.$$
      (What is the geometric meaning of this identity?)

      (iii) (5 points) Find $\|\gamma'(\theta)\|$. Hence write the arc length of $\gamma(\theta)$ as an integral, no need to solve it.
      (Hint: What are the ways to find the norm of a vector?)

   In this question, you may need to use double angle formula or product to sum formula, see the next page for reference.

Double Angle Formula

$$\begin{aligned}\sin(2x) &= 2\sin(x)\cos(x)\\ \cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= 2\cos^2(x) - 1\\ &= 1 - 2\sin^2(x)\end{aligned}$$

Product to Sum Formula

$$\begin{aligned}\sin A\cos B &= \frac{1}{2}[\sin(A+B) + \sin(A-B)]\\ \cos A\sin B &= \frac{1}{2}[\sin(A+B) - \sin(A-B)]\\ \cos A\cos B &= \frac{1}{2}[\cos(A+B) + \cos(A-B)]\\ \sin A\sin B &= \frac{1}{2}[\cos(A-B) - \cos(A+B)]\end{aligned}$$

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